Homework 3
MTH 869 Algebraic Topology

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Proposition 0.1 (Exercise 1.1.10)**.** *Let (X, x_0) and (Y, y_0) be pointed, path-connected spaces. Let $f : I \to X \times \{y_0\}$ and $g : I \to \{x_0\} \times Y$ both be loops based at (x_0, y_0). Via inclusions, we can think of f, g as loops $I \to X \times Y$ based at (x_0, y_0). Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the standard projections. Then we have $f \cdot g \simeq g \cdot f$ via the homotopy*

$$h_t(s) = \begin{cases} g(2s) & 0 \le s \le t/2 \\ (p_X f(2s - t), p_Y g(t)) & t/2 \le s \le t/2 + 1/2 \\ g(2s - 1) & t/2 + 1/2 \le s \le 1 \end{cases}$$

As a consequence, we have $[f][g] = [g][f]$.

Proof. Define h_t as above. We check that the potentially conflicting definitions agree on the overlaps. When $s = t/2$, we have

$$\begin{aligned} h_t(s) &= g(2s) = g(t) = (x_0, p_Y g(t)) \\ h_t(s) &= (p_X f(2(t/2) - t), p_Y(g(t)) = (p_X f(0), p_Y g(t)) = (x_0, p_Y g(t)) \end{aligned}$$

When $s = t/2 + 1/2$, we have

$$\begin{aligned} h_t(s) &= (p_X f(2(t/2 + 1/2) - t), p_Y g(t)) = (p_X f(t + 1 - t), p_Y g(t)) \\ &= (p_X f(1), p_Y g(t)) = (x_0, p_Y g(t)) \\ h_t(s) &= (x_0, p_Y g(2(t/2 + 1/2) - 1)) = (x_0, p_Y g(t + 1 - 1)) = (x_0, p_Y g(t)) \end{aligned}$$

Now we check that h_t is a homotopy of paths. It is immediate to check that it fixes the endpoints for all t:

$$\begin{aligned} h_t(0) &= g(0) = (x_0, y_0) \\ h_t(1) &= g(2(1) - 1) = g(1) = (x_0, y_0) \end{aligned}$$

Now we show that $h_0 = f \cdot g$ and $h_1 = g \cdot f$.

$$\begin{aligned}
h_0(s) &= \begin{cases} g(2s) & 0 \le s \le 0 \\ (p_X f(2s), p_Y g(0)) & 0 \le s \le 1/2 \\ g(2s-1) & 1/2 \le s \le 1 \end{cases} \\
&= \begin{cases} (p_X(f(2s), y_0) & 0 \le s \le 1/2 \\ g(2s-1) & 1/2 \le s \le 1 \end{cases} \\
&= \begin{cases} f(2s) & 0 \le s \le 1/2 \\ g(2s-1) & 1/2 \le s \le 1 \end{cases} \\
&= f \cdot g(s) \\
h_1(s) &= \begin{cases} g(2s) & 0 \le s \le 1/2 \\ (p_X f(2s-1), p_Y g(1)) & 1/2 \le s \le 1 \\ g(2s-1) & 1 \le s \le 1 \end{cases} \\
&= \begin{cases} g(2s) & 0 \le s \le 1/2 \\ (p_X(f(2s-1), y_0) & 1/2 \le s \le 1 \end{cases} \\
&= \begin{cases} g(2s) & 0 \le s \le 1/2 \\ f(2s-1) & 1/2 \le s \le 1 \end{cases} \\
&= g \cdot f(s)
\end{aligned}$$

Thus $f \cdot g \simeq g \cdot f$. Hence

$$[f][g] = [f \cdot g] = [g \cdot f] = [g][f]$$

□

Proposition 0.2 (Exercise 1.1.14). *Let X, Y be path connected spaces. Let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ be the projections $(x, y) \mapsto x$ and $(x, y) \mapsto y$ respectively. We have induced homomorphisms $p_{1*} : \pi_1(X \times Y) \to \pi_1(X)$ and $p_{2*} : \pi_1(X \times y) \to \pi_1(Y)$. Define $\phi : \pi_1(X \times Y) \to \pi_1(X) \times \pi_1(Y)$ by*

$$[f] \mapsto (p_{1*}[f], p_{2*}[f])$$

Then ϕ is a group isomorphism.

Proof. First we show that ϕ is a group homomorphism. Since p_{1*} is a homomorphism,

$$p_{1*}([f][g]) = (p_{1*}[f])(p_{1*}[g])$$

and likewise for p_{2*}. Thus

$$\begin{aligned}
\phi([f][g]) = (p_{1*}([f][g]), p_{2*}([f][g])) &= \big((p_{1*}[f])(p_{1*}[g]), (p_{2*}[f])(p_{2*}[g])\big) \\
&= \big(p_{1*}[f], p_{2*}[f]\big)\big(p_{1*}[g], p_{2*}[g]\big) = (\phi[f])(\phi[g])
\end{aligned}$$

so ϕ is a homomorphism. Now we show that ϕ is surjective. Let $([f_x], [f_y]) \in \pi_1(X) \times \pi_1(Y)$, and choose representatives f_x, f_y. Define $f : I \to X \times Y$ by $f(t) = (f_x(t), f_y(t))$. Then

$$\phi[f] = (p_{1*}[f], p_{2*}[f]) = ([p_1 \circ f], [p_2 \circ f]) = ([f_x], [f_y])$$

Hence ϕ is surjective. Finally, we show that ϕ is injective by showing that the kernel is trivial. Let $[f] \in \ker \phi$, and choose a representative f. Since $[f] \in \ker \phi$, we have $p_{1*}[f] = [p_1 \circ f] = 0$ and $p_{2*}[f] = [p_2 \circ f] = 0$. Thus $p_1 \circ f$ and $p_2 \circ f$ are homotopic to constant maps, say via homotopies $h_t^1 : I \to X$ and $h_t^2 : I \to Y$, that is,

$$\begin{aligned} h_0^1 &= p_1 \circ f \qquad & h_0^2 &= p_2 \circ f \\ h_1^1 &= c_1 & h_1^2 &= c_2 \end{aligned}$$

for some constants $c_1 \in X, c_2 \in Y$. Then f is homotopic to a constant map via $(s,t) \mapsto (h_t^1(s), h_t^2(s))$, since

$$\begin{aligned} (s,0) &\mapsto (h_0^1(s), h_0^2(s)) = (p_1 \circ f(s), p_2 \circ f(s)) = f(s) \\ (s,1) &\mapsto (h_1^1(s), h_1^2(s)) = (c_1, c_2) \end{aligned}$$

Thus $[f] = 0$, so $\ker \phi$ is trivial, so ϕ is injective. This completes the proof that ϕ is an isomorphism. □

Lemma 0.3 (for topological group problem)**.** *Let X be a topological group with identity e. For loops $f, g : I \to X$ based at e, define $f * g : I \to X$ by $(f * g)(s) = f(s)g(s)$. This induces $* : \pi_1(X, e) \times \pi_1(X, e) \to \pi_1(X, e)$ given by $[f] * [g] = [f * g]$. We claim that this is well-defined. Furthermore,*

$$(f * g) \cdot (f' * g') = (f \cdot f') * (g \cdot g')$$

Using $\cdot$ for the usual multiplication in $\pi_1(X)$, we have

$$\big([f] \cdot [f']\big) * \big([g] \cdot [g']\big) = \big([f] * [g]\big) \cdot \big([f'] * [g']\big)$$

Proof. Suppose $[f] = [f']$ and $[g] = [g]'$, so we have homotopies $f_t : I \to X$ and $g_t : I \to X$ satisfying

$$\begin{aligned} f_0 &= f \qquad & f_1 &= f' \qquad & f_t(0) &= f_t(1) = e \\ g_0 &= g & g_1 &= g' & g_t(0) &= g_t(1) = e \end{aligned}$$

Then define $h_t : I \to X$ by $h_t(s) = f_t(s) g_t(s)$. Then

$$\begin{aligned} h_0(s) &= f(s)g(s) = (f * g)(s) \\ h_1(s) &= f'(s)g'(s) = (f' * g')(s) \\ h_t(0) &= f_t(0)g_t(0) = e \\ h_t(1) &= f_t(1)g_t(1) = e \end{aligned}$$

Thus h_t is a homotopy from $f * g$ to $f' * g'$, so $[f * g] = [f' * g']$. Thus the operation is well-defined. Now we compute

$$
\begin{aligned}
\big((f * g) \cdot (f' * g')\big)(s) &= \begin{cases} (f * g)(2s) & 0 \le s \le 1/2 \\ (f' * g')(2s-1) & 1/2 \le s \le 1 \end{cases} \\
&= \begin{cases} f(2s)g(2s) & 0 \le s \le 1/2 \\ f'(2s-1)g'(2s-1) & 1/2 \le s \le 1 \end{cases} \\
&= \left(\begin{cases} f(2s) & 0 \le s \le 1/2 \\ f'(2s-1) & 1/2 \le s \le 1 \end{cases}\right) \left(\begin{cases} g(2s) & 0 \le s \le 1/2 \\ g'(2s-1) & 1/2 \le s \le 1 \end{cases}\right) \\
&= \big(f \cdot f'(s)\big)\big(g \cdot g'(s)\big) \\
&= \big((f \cdot f') * (g \cdot g')\big)(s)
\end{aligned}
$$

Thus

$$(f * g) \cdot (f' * g') = (f \cdot f') * (g \cdot g')$$

By reflexivity, from this we get

$$(f * g) \cdot (f' * g') \simeq (f \cdot f') * (g \cdot g')$$

Thus

$$\big([f] \cdot [f']\big) * \big([g] \cdot [g']\big) = \big([f] * [g]\big) \cdot \big([f'] * [g']\big)$$

□

Lemma 0.4 (Eckmann-Hilton Theorem). *Let X be a set with two binary operations $*, \cdot$. Suppose that both operations have a unit, that is, there exist $e, e' \in X$ so that*

$$e \cdot x = x = x \cdot e \qquad e' * x = x = x * e'$$

for all $x \in X$. Suppose also that for all $w, x, y, z \in X$ we have

$$(w \cdot x) * (y \cdot z) = (w * y) \cdot (x * z)$$

Then $\cdot, *$ are equal, associative, and commutative. That is, for all $x, y, z \in X$,

$$
\begin{aligned}
x \cdot y &= x * y \\
x \cdot y &= y \cdot x \\
(x \cdot y) \cdot z &= x \cdot (y \cdot z)
\end{aligned}
$$

Proof. First, we show that $e = e'$.

$$e = e \cdot e = (e' * e) \cdot (e * e') = (e' \cdot e) * (e \cdot e') = e' * e' = e'$$

Let $x, y \in X$. Then

$$x \cdot y = (x * e) \cdot (e * y) = (x \cdot e) * (e \cdot y) = x * y$$

Thus the operations coincide. Also,

$$x \cdot y = (e * x) \cdot (y * e) = (e \cdot y) * (x \cdot e) = y * x$$

Thus $x * y = y * x$ so the operations are commutative. Finally,

$$x \cdot (y \cdot z) = (x \cdot 1) \cdot (y \cdot z) = (x \cdot y) \cdot (1 \cdot z) = (x \cdot y) \cdot z$$

Thus they are associative. □

Proposition 0.5 (written exercise from Prof. Hedden)**.** *Let X be a topological group. Then $\pi_1(X, e)$ is abelian.*

Proof. Define multiplication of paths elementwise as in Lemma 0.3 above. As shown in that lemma,

$$\big([f] \cdot [f']\big) * \big([g] \cdot [g']\big) = \big([f] * [g]\big) \cdot \big([f'] * [g']\big)$$

Thus $\cdot, *$ are binary operations on $\pi_1(X)$ satisfying the hypotheses of the Eckmann-Hilton Theorem, so they are equal and abelian. Thus the usual multiplication on $\pi_1(X)$ is abelian. □

Proposition 0.6 (Exercise 1.1.18, part one)**.** *Let A be a path-connected space. Form X by attaching an n-cell e^n with $n \geq 2$. Then the inclusion $\iota : A \hookrightarrow X$ induces a surjection on π_1. That is, $\iota_* : \pi_1(A) \to \pi_1(X)$ is surjective.*

Proof. Let $f : S^{n-1} \to A$ be the attaching map. Then $X = A \cup e^n$, where A and e^n are path connected and open in X, and $A \cap e^n = f(S^{n-1})$ is also path-connected. Let $x_0 \in f(S^{n-1})$. By Lemma 1.15 (Hatcher), every loop in X based at x_0 is homotopic to a product of loops, where each loop is either contained in e^n or A. Since $n \geq 2$, a loop contained in e^n is nullhomotopic, so every loop in X is homotopic to a loop in A. Thus if $[f] \in \pi_1(X, x_0)$, there there is a loop $f' : I \to A$ so that $[f'] = [f]$. We have $f' = \iota \circ f'$, so

$$\iota_*[f'] = [\iota \circ f'] = [f'] = [f]$$

Hence ι_* is surjective. □

Proposition 0.7 (Exercise 1.1.18a)**.** *The wedge sum $S^1 \vee S^2$ has fundamental group $\mathbb{Z}$.*

Proof. As noted in Example 0.11 of Hatcher, $S^1 \vee S^2$ can be formed by attaching S^2 to S^1 via a constant map. By the above, the inclusion $\iota : S^1 \to S^1 \vee S^2$ induces a surjection $\iota_* : \pi_1(S^1) \to \pi_1(S^1 \vee S^2)$. By the first isomorphism theorem of groups,

$$\pi_1(S^1 \vee S^2) \cong \pi_1(S^1)/\ker \iota_*$$

Thus $\pi_1(S^1 \vee S^2)$ is isomorphic to a quotient group of $\mathbb{Z}$, so it is cyclic. Note that $\pi_1(S^1 \vee S^2)$ is not finite, since it contains infinitely many non-homotopic loops (take loops winding n times around the S^1 part for $n \in \mathbb{N}$). Thus $\pi_1(S^1 \vee S^2)$ is infinite cyclic, that is, isomorphic to $\mathbb{Z}$. □

Proposition 0.8 (Exercise 1.1.18b)**.** *Let X be a path-connected CW complex with X^1 its 1-skeleton. Then the inclusion map $\iota : X^1 \hookrightarrow X$ induces a surjection $\iota_* : \pi_1(X^1) \to \pi_1(X)$.*

Proof. The space X is formed from X^1 by attaching n-cells for $n \geq 2$. First, suppose there are finitely many cells $e_1, \dots, e_k$. Let $X_0 = X^1$, and define X_i to be the CW complex formed after attaching e_i to X_{i+1}, so $X_k = X$. Then each inclusion $\iota_i : X_i \hookrightarrow X_{i+1}$ induces a surjection $\iota_{i*} : \pi_1(X_i) \to \pi_1(X_{i+1})$, so the (finite) composition

$$\iota_{k*}\iota_{(k-1)*} \dots \iota_{1*}\iota_{0*} = (\iota_k \iota_{k-1} \dots \iota_1 \iota_0)_* = \iota_*$$

is surjective.

Now suppose that X has infinitely many cells. Let $[f] \in \pi_1(X)$ and choose a representative loop f. The image of f is a compact subset of X, so by Proposition A.1 in the Appendix (Hatcher), the image is contained in a finite subcomplex $\overline{X} \subset X$. Let $\overline{X}^1$ be the 1-skeleton of $\overline{X}$, so $\overline{X}^1 \subset X^1$. Let $\bar{\iota} : \overline{X}^1 \hookrightarrow \overline{X}$ be the inclusion.

By the result in the finite case, $\bar{\iota}_* : \pi_1(\overline{X}^1) \to \pi_1(\overline{X})$ is surjective. Since the image of f lies in $\overline{X}$, we can think of $[f] \in \pi_1(\overline{X})$. By surjectivity of $\bar{\iota}_*$, there exists $\left[\widetilde{f}\right] \in \pi_1(\overline{X}^1)$ so that $\bar{\iota}_* \left[\widetilde{f}\right] = [f]$. Choose a representative $\widetilde{f}$. Since the image of $\widetilde{f}$ is contained in $\overline{X}^1 \subset X^1$, we also have $\left[\widetilde{f}\right] \in \pi_1(X^1)$. Note that $\iota \circ \widetilde{f} = \bar{\iota} \circ \widetilde{f}$ since the image of $\widetilde{f}$ lies in $\overline{X}^1$. Thus

$$\iota_* \left[\widetilde{f}\right] = \left[\iota \circ \widetilde{f}\right] = \left[\bar{\iota} \circ \widetilde{f}\right] = \bar{\iota}_* \left[\widetilde{f}\right] = [f]$$

Thus ι_* is surjective. □

Proposition 0.9 (Exercise 1.3.1)**.** *Let $p : \widetilde{X} \to X$ be a covering map and $A \subset X$, and let $\widetilde{A} = p^{-1}(A)$. Then the restriction $p|_{\widetilde{A}} : \widetilde{A} \to A$ is a covering map.*

Proof. Let $a \in A$. Since p is a covering map, there is an evenly covered neighborhood U so that $a \in U \subset X$. That is,

$$p^{-1}(U) = \bigsqcup_\alpha U_\alpha$$

where each U_α is mapped homeomorphically to U by p. By definition of subspace topology, $U \cap A$ is an open neighborhood of a in A. Using basic properties of preimages,

$$p^{-1}(U \cap A) = p^{-1}(U) \cap p^{-1}(A) = \left(\bigsqcup_\alpha U_\alpha\right) \cap \widetilde{A} = \bigsqcup_\alpha (U_\alpha \cap \widetilde{A})$$

Then

$$p(U_\alpha \cap \widetilde{A}) \subset p(U_\alpha) \cap p(\widetilde{A}) = p(U_\alpha) \cap A = U \cap A$$

so $p|_{U_\alpha \cap \widetilde{A}} : U_\alpha \cap \widetilde{A} \to U \cap A$ is well-defined. First we claim that it is surjective. Let $x \in U \cap A$. Since $x \in U$, there exists $x_\alpha \in U_\alpha$ so that $p(x_\alpha) = x$., Since $x \in A$, $x_\alpha \in \widetilde{A}$, so $x_\alpha \in U_\alpha \cap \widetilde{A}$. Thus $p|_{U_\alpha \cap \widetilde{A}}$ is surjective. It is injective since it is a restriction of the injective map $p|_{U_\alpha} : U_\alpha \to U$. Thus it is bijective. It is continuous because it is a restriction of the continuous map p. It has a continuous inverse since its inverse is a restriction of the continuous inverse of $p|_{U_\alpha}$.

Thus $p|_{U_\alpha \cap \widetilde{A}}$ maps $U_\alpha \cap \widetilde{A}$ homeomorphically onto $U \cap A$. Since $U \cap A$ is a disjoint union of such sets, $U \cap A$ is an evenly covered neighborhood of $a \in A$. Hence $p|_{\widetilde{A}}$ is a covering map. □

Lemma 0.10 (for Exercise 1.3.2)**.** *Let $p_1 : \widetilde{X}_1 \to X_1$ and $p_2 : \widetilde{X}_2 \to X_2$ be set maps. Let $U_1 \subset X_1$ and $U_2 \subset X_2$. Then*

$$(p_1 \times p_2)^{-1}(U_1 \times U_2) = p_1^{-1}(U_1) \times p_2^{-1}(U_2)$$

Proof. This is a straightforward use of definitions.

$$\begin{aligned}(p_1 \times p_2)(U_1 \times U_2) &= \{(\widetilde{x}_1, \widetilde{x}_2) \in \widetilde{X}^1 \times \widetilde{X}^2 : (p_1 \times p_2)(\widetilde{x}_1, \widetilde{x}_2) \in U_1 \times U_2\} \\ &= \{(\widetilde{x}_1, \widetilde{x}_2) \in \widetilde{X}^1 \times \widetilde{X}^2 : (p_1(\widetilde{x}_1), p_2(\widetilde{x}_2)) \in U_1 \times U_2\} \\ &= \{(\widetilde{x}_1, \widetilde{x}_2) \in \widetilde{X}^1 \times \widetilde{X}^2 : \widetilde{x}_1 \in p_1^{-1}(U_1), \widetilde{x}_2 \in p_2^{-1}(U_2)\} \\ &= p_1^{-1}(U_1) \times p_2^{-1}(U_2)\end{aligned}$$

□

Proposition 0.11 (Exercise 1.3.2)**.** *Let $p_1 : \widetilde{X}_1 \to X_1$ and $\widetilde{p}_2 : \widetilde{X}_2 \to X_2$ be covering maps. Then the product $p_1 \times p_2 : \widetilde{X}_1 \times \widetilde{X}_2 \to X_1 \times X_2$ is a covering map.*

Proof. Let $(x_1, x_2) \in X_1 \times X_2$. Since p_1, p_2 are covering maps, there exist evenly covered neighborhoods U_1 of x_1 and U_2 of x_2. That is,

$$p_1^{-1}(U_1) = \bigsqcup_{\alpha \in A} U_1^\alpha \qquad p_2^{-1}(U_2) = \bigsqcup_{\beta \in B} U_2^\beta$$

where p_1 maps each U_1^α homeomorphically to U_1 and p_2 maps each U_2^β homeomorphically to U_2. By definition of the product topology, $U_1 \times U_2$ is an open neighborhood of (x_1, x_2). Using the previous lemma,

$$(p_1 \times p_2)^{-1}(U_1 \times U_2) = p_1^{-1}(U_1) \times p_2^{-1}(U_2) = \left(\bigsqcup_{\alpha \in A} U_1^\alpha\right) \times \left(\bigsqcup_{\beta \in B} U_2^\beta\right) = \bigsqcup_{(\alpha,\beta) \in A \times B} (U_1^\alpha \times U_2^\beta)$$

Again, by definition of the product topology, $U_1^\alpha \times U_2^\beta$ is open in $\widetilde{X}_1 \times \widetilde{X}_2$. We claim that $(p_1 \times p_2)|_{U_1^\alpha \times U_2^\beta} : U_1^\alpha \times U_2^\beta \to U_1 \times U_2$ is a homeomorphism.

First we show that it is surjective. Let $(x_1, x_2) \in U_1 \times U_2$. Then by the even covering properties of p_1, p_2, there exist $\widetilde{x}_1 \in U_1^\alpha$ and $\widetilde{x}_2 \in U_2^\beta$ so that $p_1(\widetilde{x}_1) = x_1$ and $p_2(\widetilde{x}_2) = x_2$.

Thus $(\widetilde{x}_1, \widetilde{x}_2) \in U_1^\alpha \times U_2^\beta$ and $(p_1 \times p_2)(\widetilde{x}_1, \widetilde{x}_2) = (x_1, x_2)$. This establishes surjectivity. Now suppose that there exist $(\widetilde{x}_1, \widetilde{x}_2), (\widetilde{y}_1, \widetilde{y}_2) \in U_1^\alpha \times U_2^\beta$ so that

$$(p_1 \times p_2)(\widetilde{x}_1, \widetilde{x}_2) = (p_1 \times p_2)(\widetilde{y}_1, \widetilde{y}_2)$$

Then $p_1(\widetilde{x}_1) = p_1(\widetilde{y}_1)$ with $\widetilde{x}_1, \widetilde{y}_1 \in U_1^\alpha$. But then by injectivity of p_1 on this domain, this implies $\widetilde{x}_1 = \widetilde{y}_1$. Similarly, $\widetilde{x}_2 = \widetilde{y}_2$. Hence $p_1 \times p_2$ is injective on this domain. Continuity of $p_1 \times p_2$ comes from the properties of the product topology, and continuity of the inverse on the restricted domain also comes from the properties of the product topology, this time on $U_1^\alpha \times U_2^\beta$.

Thus $(p_1 \times p_2)|_{U_1^\alpha \times U_2^\beta} : U_1^\alpha \times U_2^\beta \to U_1 \times U_2$ is a homeomorphism. Thus $U_1 \times U_2$ is evenly covered, so $p_1 \times p_2$ is a covering map. □

Proposition 0.12 (written exercise from Prof. Hedden)**.** *Not every local homeomorphism is a covering map.*

Proof. Let $X = \{x_1, x_2\}$ with the trivial topology, that is, the only open sets are $\emptyset, X$. Let $Y = \{y_1, y_2, y_3\}$ with open sets $\emptyset, Y, \{y_1, y_2\}$. (Note that this is a topology on Y.) Define $f : X \to Y$ by $f(x_1) = y_1$ and $f(x_2) = y_2$. We claim that f is a local homeomorphism but not a covering map. First, note that f is continuous, since the preimages

$$f^{-1}\emptyset = \emptyset \qquad f^{-1}(Y) = X \qquad f^{-1}(\{y_1, y_2\}) = X$$

are open. It is a local homeomorphism because for either $x_1, x_2 \in X$, neighborhood $U = X$ gives the restriction $f : \{x_1, x_2\} \to \{y_1, y_2\}$, which is a homeomorphism. However, f is not a covering map. Consider the point $y_3 \in Y$. The only open neighborhood is Y itself, but the preimage of Y is X, which is a disjoint union of spaces mapped homeomorphically to X. □